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ON EQUATIONS FOR CYLINDRICAL SHELLS: MORLEY AND FLÜGGE

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Abstract *The effect of simplifications introduced into the theory of elastic linear cylindrical shells is studied comparing two classical shell theories. And a numerical comparison is provided by the analysis of a cylindrical shell intersection problem.*
A computational model is developed to study the intersection of two inclined circular cylindrical shells via a Fourier series approach.
The analysis using equations from Flügge and Morley reveals some of the effects that the simplifications introduced by Morley have on the stress distribution in the shells. An alternative form to Morley's equations is presented and the stress distributions resulting from implementing the suggested form is shown.
Key Words: *Shell, Flügge, Morley*

1. INTRODUCTION

The first set of equations for the theory of thin-elastic shells was proposed by Love in 1888. It was not a perfect theory. In fact, later it was found that the equations proposed by Love had some inconsistencies related to the rigid-body modes.

By 1932, Donnell published a set of simple equations for the theory of linear shells. The aesthetic of Donnell's equations immediately made them very popular. However, the shortcomings and limitations of Donnell's equations soon started to appear.

Also in 1932 another theory of shells was proposed. Flügge used Love-Kirchhoff assumptions to derive a set of equations which are considered the most precise equations of the linear theory. Unfortunately, the complexity of the equations brought resistance against their use.

During the subsequent years several equations were proposed. The approach was to construct a reliable and precise set of equations with the aesthetics of Donnell's equations.

Morley studied Donnell's equations and effectively removed some of the inconsistencies in those equations. In 1958, he proposed a set of equations that resembles the elegance of Donnell's equations but as he claims they retrieve the precision of Flügge's equations.

The subject of the present study is to verify Morley's claim. And at the same time illuminate some details concerning the effect of some terms left out in Morley's formulation.

2. CYLINDRICAL INTERSECTION

Let us consider the equilibrium equations of the cylindrical shells in the displacement form, as presented by the theories of: Flügge, Morley and Donnell. The equations are presented in dimensionless form, where $\{\bar{u}, \bar{v}, \bar{w}\}$ are the displacements along the coordinate directions and $\frac{\partial(\,)}{\partial x} = (\;)\prime$ and $\frac{\partial(\,)}{\partial \theta} = (\;)^{\cdot}$ are the corresponding partial derivatives.

- **Equilibrium Equations: Flügge's Displacement Form**

$$\bar{u}'' + \tfrac{1}{2}(1-\nu)\bar{u}^{\cdot\cdot} + \tfrac{1}{2}(1+\nu)\bar{v}'^{\cdot} + \nu\bar{w}' + k[\tfrac{1}{2}(1-\nu)\bar{u}^{\cdot\cdot} - \bar{w}''' + \tfrac{1}{2}(1-\nu)\bar{w}'^{\cdot\cdot}] = 0$$

$$\tfrac{1}{2}(1+\nu)\bar{u}'^{\cdot} + \tfrac{1}{2}(1-\nu)\bar{v}'' + \bar{v}^{\cdot\cdot} + w^{\cdot} + k[\tfrac{3}{2}(1-\nu)\bar{v}'' - \tfrac{1}{2}(3-\nu)\bar{w}'^{\cdot\cdot}] = 0$$

$$\nu\bar{u}' + \bar{v}^{\cdot} + \bar{w} + k[\tfrac{1}{2}(1-\nu)\bar{u}'^{\cdot\cdot} - \bar{u}''' - \tfrac{1}{2}(3-\nu)\bar{v}''^{\cdot} + \nabla^4\bar{w}2\bar{w}^{\cdot\cdot} + \bar{w}] = 0 \tag{1}$$

- **Equilibrium Equations: Donnell's Displacement Form**

$$\bar{u}'' + \tfrac{1}{2}(1-\nu)\bar{u}^{\cdot\cdot} + \tfrac{1}{2}(1+\nu)\bar{v}'^{\cdot} + \nu\bar{w}' = 0$$

$$\tfrac{1}{2}(1+\nu)\bar{u}'^{\cdot} + \tfrac{1}{2}(1-\nu)\bar{v}'' + \bar{v}^{\cdot\cdot} + w^{\cdot} - k[(\nabla^2\bar{w})^{\cdot}] = 0 \tag{2}$$

$$\nu\bar{u}' + \bar{v}^{\cdot} + \bar{w} + k[\nabla^4\bar{w}] = 0$$

- **Equilibrium Equations: Morley's Displacement Form (Complete Form)**

$$\bar{u}'' + \tfrac{1}{2}(1-\nu)\bar{u}^{\cdot\cdot} + \tfrac{1}{2}(1+\nu)\bar{v}'^{\cdot} + \nu\bar{w}' = 0$$

$$\tfrac{1}{2}(1+\nu)\bar{u}'^{\cdot} + \tfrac{1}{2}(1-\nu)\bar{v}'' + \bar{v}^{\cdot\cdot} + w^{\cdot} - k[(\nabla^2\bar{w})^{\cdot} + \bar{w}^{\cdot} + (1-\nu)\bar{v}''] = 0$$

$$\nu\bar{u}' + \bar{v}^{\cdot} + \bar{w} + k[\nabla^4\bar{w} - \bar{u}''' - \tfrac{1}{2}(3-\nu)\bar{v}''^{\cdot}] = 0 \tag{3}$$

- **Equilibrium-Displacement Equations**

Donnell
$$\begin{cases} \nabla^4 u = \nu w_{,xxx} - w_{,x\theta\theta} \\ \nabla^4 v = (2+\nu) w_{,xx\theta} + w_{,\theta\theta\theta} \\ \nabla^8 w + 4\hat{k}^4 w_{,xxxx} = 0 \end{cases}$$

Morley
Simplified Form
$$\begin{cases} \nabla^4 u = \nu w_{,xxx} - w_{,x\theta\theta} \\ \nabla^4 v = (2+\nu) w_{,xx\theta} + w_{,\theta\theta\theta} \\ \nabla^4(\nabla^2 + 1)^2 w + 4\hat{k}^4 w_{,xxxx} = 0 \end{cases}$$

[1]

Morley's Displacement Complete Form as we present above was not published by Morley in his paper. Rather the final form, the simplied one was presented. In order to obtain the simplied form, Morley eliminated some terms of " less importance ". The point of discussion is the effect those terms may have on the numerical results.

3. CYLINDRICAL INTERSECTIONS

Figure 1 presents the intersection of two cylinders. It will be used as the model for analysis. The intersection curve Γ_1 is in a symmetry plane and Γ_2 is a curve far away from the intersection. For the case of the intersection of cylinders with the same material and dimensions, the problem can be simplified using symmetry. For such a case the analysis of one cylinder is enough. For the general case, the properties of both cylinders need to be taken into account.

Figure 2 shows the reference directions of the symmetry plane defined along the intersection curve Γ_1, and the angle β^* between the normal to the symmetry plane and the axis of the cylinder.

The problem of the cylindrical intersection can be stated in terms of finding the generalized displacements and generalized forces $\{\mathbf{d}, \mathbf{f}\}$ everywhere in the cylinders, provided that the following conditions are satisfied along the boundary curve:

$$\begin{cases} \mathbf{d_{1s}} - \mathbf{d_{2s}} = 0 & \text{Displacement Continuity} \\ \mathbf{f_{1s}} + \mathbf{f_{2s}} = 0 & \text{Force Equilibrium} \end{cases} \tag{4}$$

where the sub-indices $(.)_{\mathbf{1s}}, (.)_{\mathbf{2s}}$ refers to components along the directions of the symmetry plane for each cylinder as shown in Figure 2. The generalized forces are effective forces along the boundary curve. For such a case, four components are required for the generalized forces as well as for the generalized displacements.

4. RESULTS AND CONCLUSIONS

We present some numerical comparisons obtained using different set of equations. For the moment, only two examples will be shown. The first example, Figure 3, refers to the normal stresses along the intersection curve of two cylinders forming a 10 degree angle, under

[1] Note: $4\hat{k}^4 = 12(1-\nu^2)(\frac{R}{t})^2$ $\quad k = \frac{t^2}{12R^2}$, $D_s = \frac{Et}{(1-\nu^2)}$

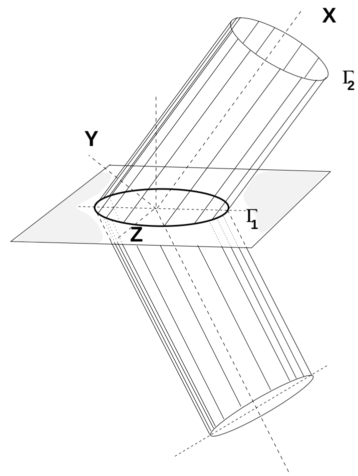

Figure 1: Cylindrical Intersection

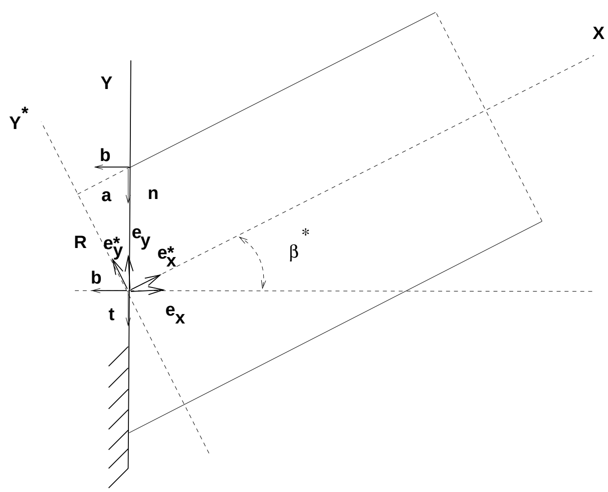

Figure 2: Model of Intersection

constant internal pressure p. The numerical solutions using Morley's simplified form practically retrieves the accuracy of Morley's complete set of equations and Flügge's equations. The same conclusion can be observed concerning the hoop stresses along the intersection.

The second example, Figure 5, refers to the normal stresses along a 20 degree angle intersection. It can be observed that there is considerable difference in the results. The numerical solution using the simplified form (S.Morley) diverges considerably when compared with the results from Flügge's equations. However, Morley's equations in the complete form (C.Morley) produce a numerical solution of the same order of accuracy of Flügge's solution. The stresses are presented in a normalized form Sn/StNom=$\frac{\sigma_n}{\frac{pR}{t}}$ where σ_n is the normal stress.

The comparision of the hoop forces along the intersection can be observed in Figure 4 and Figure 6.

Certainly, Morley's equations in the complete form are much more complex and for sure not so attractive. We recommend the use of Flügge's equations in order to avoid serious loss of precision.

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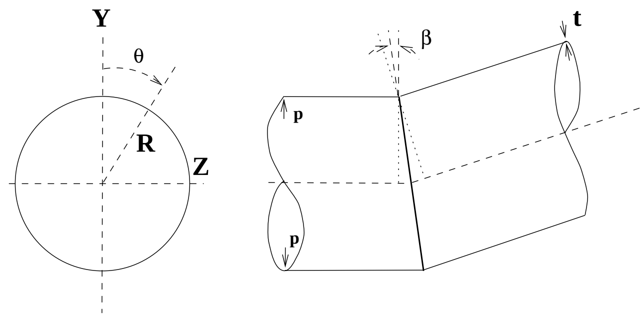


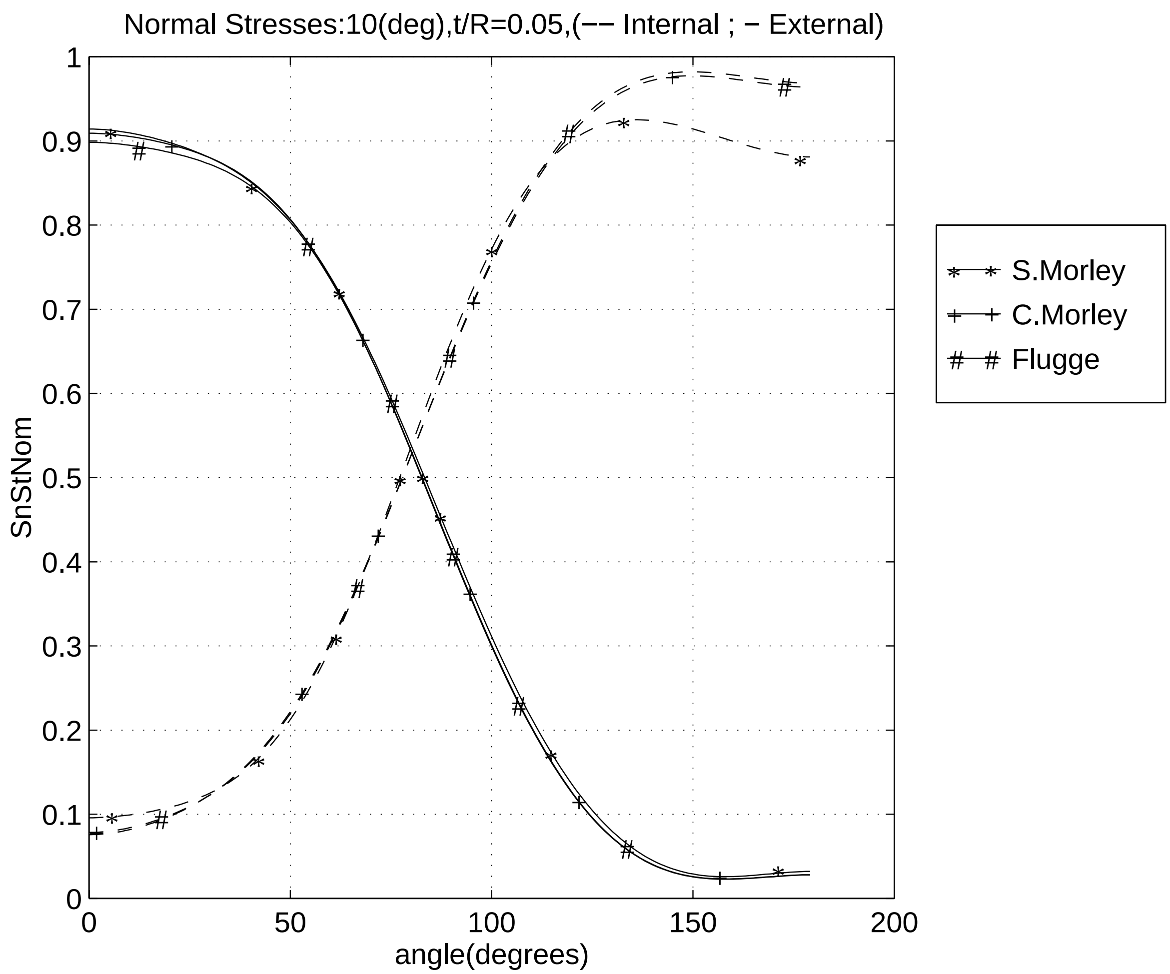
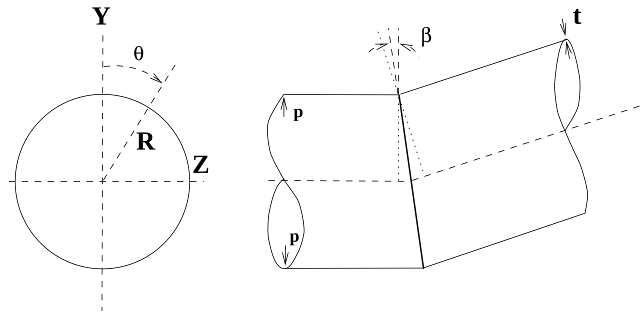


Figure 3: Normal Stresses: Morley-Flügge Mitre Angle=10(deg) t/R=0.05, Internal Pressure

- Note:
  - Number of Harmonics: 8
  - Sn/StNom= $\frac{\sigma_n}{\frac{pR}{t}}$



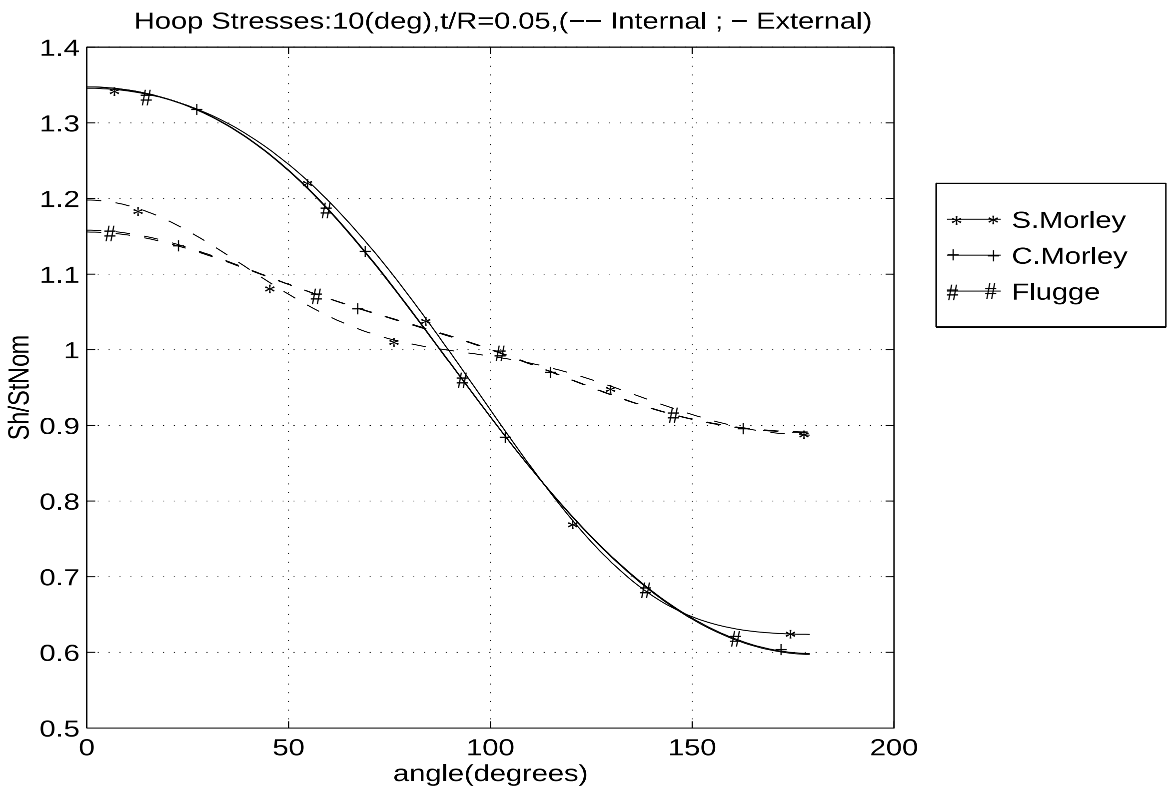
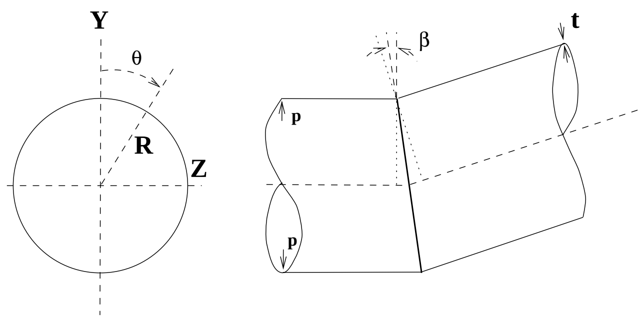


Figure 4: Hoop Stresses: Morley-Flügge Mitre Angle=10(deg) t/R=0.05, Internal Pressure

- Note:
  - Number of Harmonics: 8
  - Sh/StNom= $\frac{\sigma_h}{\frac{pR}{t}}$



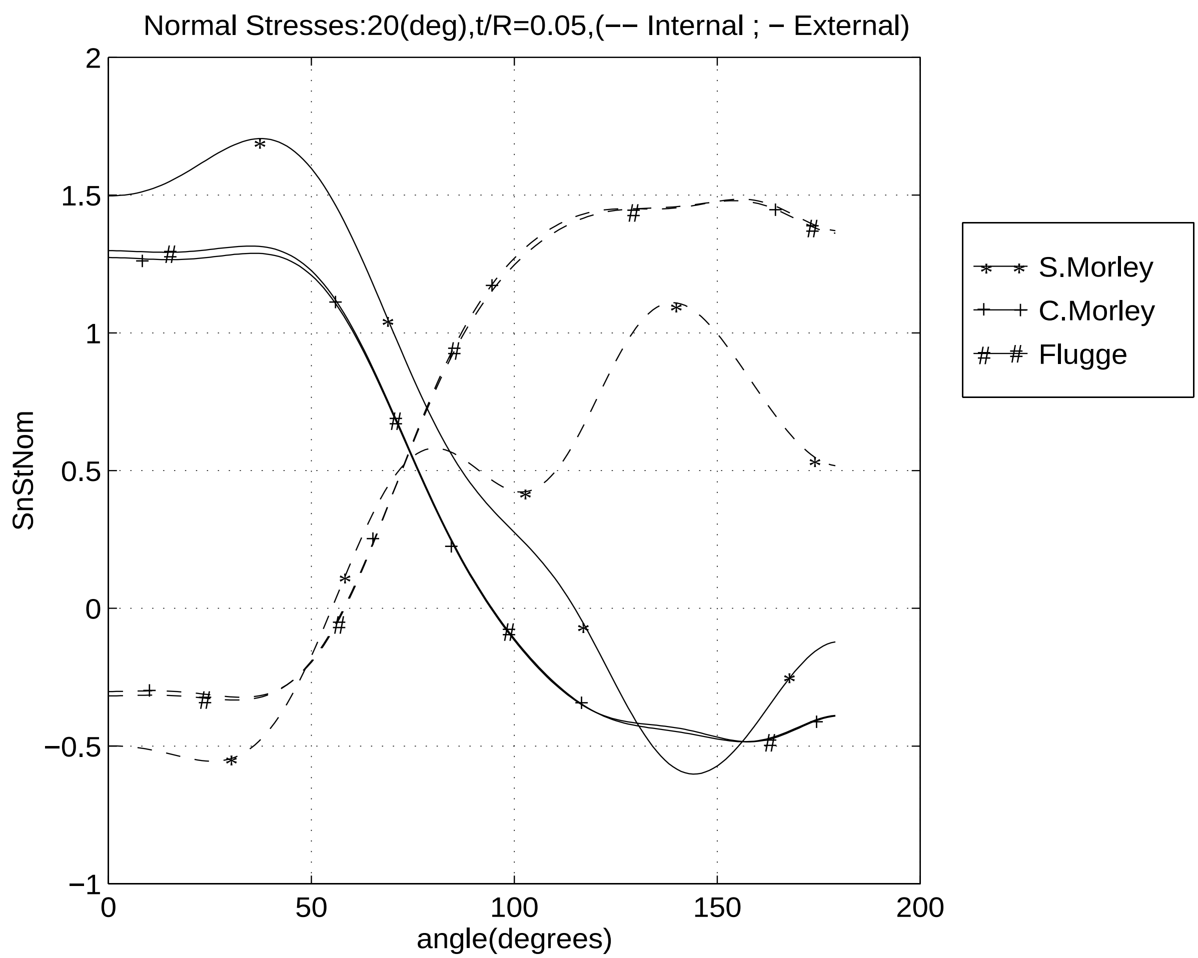


Figure 5: Normal Stresses: Morley-Flügge Mitre Angle=20(deg) t/R=0.05, Internal Pressure

- Note:
  - Number of Harmonics: 8
  - Sn/StNom= $\frac{\sigma_n}{\frac{pR}{t}}$

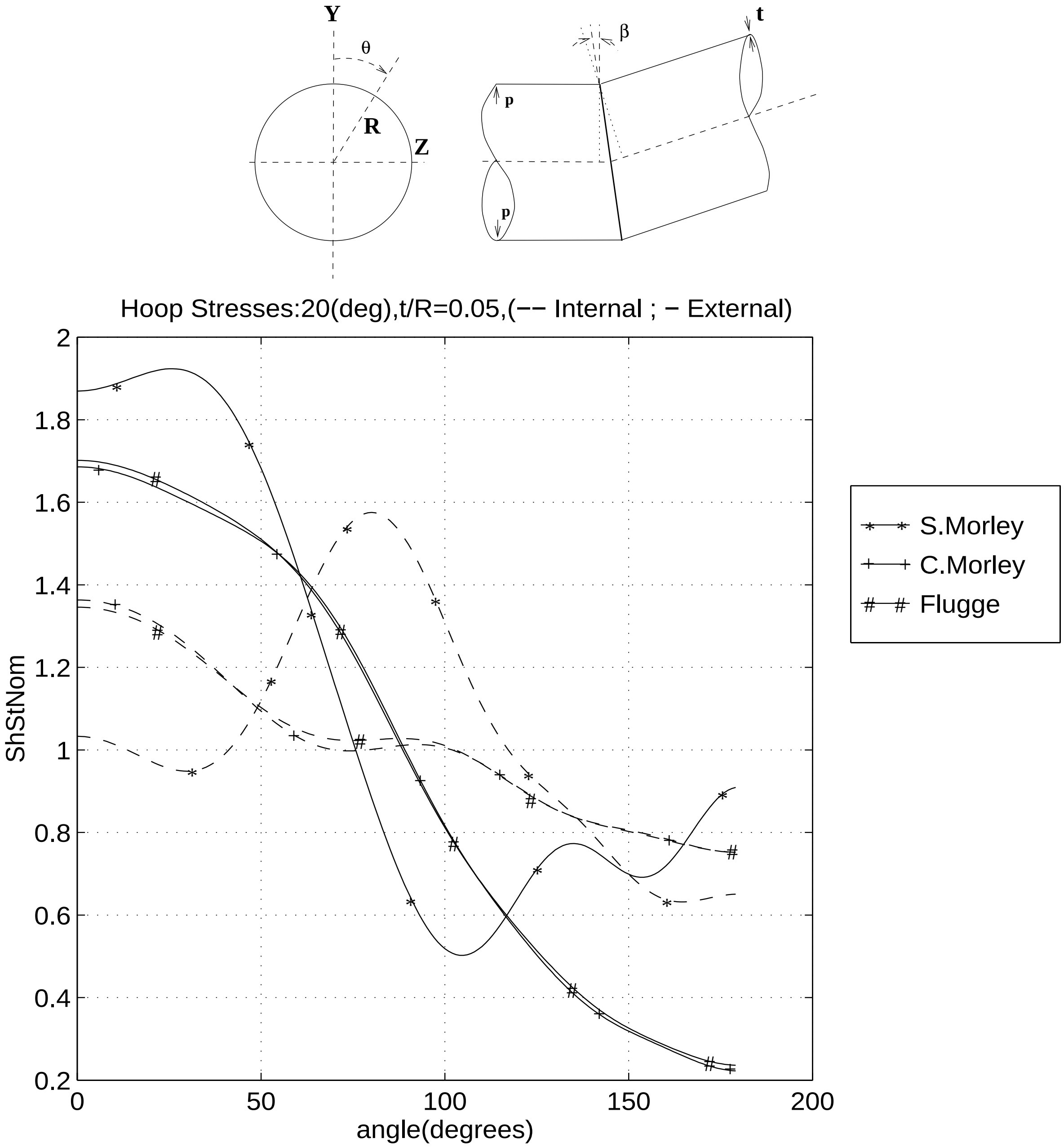


Figure 6: Hoop Stresses: Morley-Flügge Mitre Angle=20(deg) t/R=0.05, Internal Pressure

- Note:
  - Number of Harmonics: 8
  - Sh/StNom= $\frac{\sigma_h}{\frac{pR}{t}}$